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A general class of Bernstein-like bases

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Abstract

This paper presents a unified approach to deal with spaces containing simultaneously algebraic and trigonometric or hyperbolic polynomials. Bases with optimal shape preserving and stability properties are constructed. Evaluation and subdivision algorithms for them are provided. Bases for the corresponding and mixed spline spaces are also constructed. Some nice properties of these bases and the generated curves are shown.

Keywords: Shape preserving representations; Evaluation and subdivision algorithms; Trigonometric and hyperbolic functions; Splines; Extended Chebyshev spaces

1. Introduction

The construction and analysis of new function spaces that are more flexible than polynomials but with the same nice structural properties, constitutes an interesting new research trend in CAD (see [1–14]). This paper presents a unified approach to deal with these spaces for design purposes. In addition to the well known case of algebraic polynomials, our approach includes as particular examples the spaces $\langle 1, \ldots, t^{n-2}, \cosh(wt), \sinh(wt)\rangle$, and the spaces $\langle 1, \ldots, t^{n-2}, \cos(wt), \sin(wt)\rangle$, for $w \neq 0$ and $n > 1$, and piecewise functions composed of different spaces. As far as we know, these spline spaces have not yet been considered in the literature, and they permit us to deal with the complex geometry of real objects.

A system $(u_0, \ldots, u_n)$ of functions on $I \subseteq \mathbf{R}$ is normalized if $\sum_{i=0}^{n} u_i(t) = 1\ \forall t \in I$. Many shape preserving properties are obtained (see ([15–17]) when the normalized system is totally positive (that is, all the minors of all its collocation matrices are nonnegative). In contrast to the space of algebraic polynomials, which possesses normalized totally positive bases on any compact interval, there are other spaces such as the space of the trigonometric polynomials (see [10]) or the spaces $\langle 1, \ldots, t^{n-2}, \cos(wt), \sin(wt)\rangle$, $n > 1$ mentioned above, that have no normalized totally positive bases on any compact interval. Finding domain intervals where we can guarantee the existence of shape preserving representations is one of the main tasks to carry out when dealing with these spaces. Another crucial task consists of finding properties of the spaces resembling the good properties of the Bernstein basis. When

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the space has a normalized totally positive basis, Bernstein-like bases are provided by the normalized B-bases, which present optimal shape preserving properties (see [16,18,17]). A totally positive basis is called a *B-basis* if $\inf\{b_i(t)/b_j(t) \mid t \in I, b_j(t) \neq 0\} = 0$, for all $i \neq j$. In [18] it was proved that a space with a normalized totally positive basis has a unique normalized B-basis. The normalized B-basis of the space of polynomials of degree at most k on a compact interval $[a, b]$ is the corresponding Bernstein basis (see [16]). The B-spline basis is the normalized B-basis of the space of polynomial splines (see [18]).

In Section 2, we provide a unified and recursive procedure to obtain the normalized B-basis of spaces of dimension greater than 2 considered in this paper, and we call them generalized Bernstein bases. In Section 3 we show other properties of these bases, such as optimal stability, symmetry and a degree-raising formula. Section 4 shows how the de Casteljau algorithm for the evaluation of Bézier curves can be generalized to curves generated in these spaces and analyses subdivision properties. Finally, in Section 5 we consider piecewise functions, including the new and useful case in which they are composed by different spaces.

2. Generalized Bernstein bases

A space of functions $\mathcal{U} \subseteq \mathcal{C}(\mathbf{R})$ is invariant under translations if, for any $u \in \mathcal{U}$, $\tau \in \mathbf{R}$, the function $v(t) := u(t-\tau)$, $t \in \mathbf{R}$, belongs to $\mathcal{U}$. If $\mathcal{U}$ is a finite dimensional space of $\mathcal{C}^1(\mathbf{R})$ which is invariant under translations, then we have, for any $u \in \mathcal{U}$ that $h^{-1}(u(t+h) - u(t)) \in \mathcal{U}$ and, taking $h \to 0$, we can deduce that the derivative $u' \in \mathcal{U}$. Therefore $\mathcal{U} \subseteq \mathcal{C}^\infty(\mathbf{R})$.

It is well known that the solutions of a homogeneous linear differential equation of order $n + 1$

$$u^{(n+1)}(t) + a_n(t)u^{(n)}(t) + \cdots + a_0(t)u(t) = 0, \quad t \in I, \tag{1}$$

are invariant under translations if and only if the coefficients $a_i(t)$ are constant.

A space of functions $\mathcal{U} \subseteq \mathcal{C}(\mathbf{R})$ is invariant under reflections if, for any $u \in \mathcal{U}$ and $\tau \in \mathbf{R}$, the function $v(t) := u(\tau - t)$, $t \in \mathbf{R}$ belongs to $\mathcal{U}$. If $\mathcal{U}$ is invariant under reflections, then $\mathcal{U}$ is invariant under translations. If $\mathcal{U}$ is a finite dimensional space of $\mathcal{C}^1(\mathbf{R})$ which is invariant under reflections, then $\mathcal{U}$ is the set of solutions of a homogeneous linear differential equation of order $n + 1$ (1) such that its characteristic polynomial $p(t) = x^{n+1} + a_n x^n + \cdots + a_0$ is either an even or odd function.

Let I be an interval of the real line. Let us recall that an *extended Chebyshev space* is a $(n + 1)$-dimensional subspace $\mathcal{U}$ of $C^n(I)$ such that each nonzero function in $\mathcal{U}$ has at most n zeroes, counting multiplicities. Let us also recall that a basis $(u_0, \ldots, u_n)$ of an $(n + 1)$-dimensional space $\mathcal{U}$ in $\mathcal{C}^n(I)$ is *canonical* at $t \in I$ if the Wronskian matrix

$$W(u_0, \ldots, u_n)(t) := (u_j^{(i)}(t))_{0 \le i,j \le n}$$

is lower triangular with nonzero diagonal entries.

The following auxiliary result is a direct consequence of Proposition 3.2 of [1], and will be used in Theorem 2.

Lemma 1. *Let $\mathcal{U}$ be a 2-dimensional space of differentiable functions which is invariant under reflections. Let (u_0, u_1) be a canonical basis at 0 of $\mathcal{U}$ such that $W(u_0, u_1)(0)$ has positive diagonal entries. Then $\mathcal{U}$ is an extended Chebyshev space on each interval $[0, \alpha]$ with $0 < \alpha < z_{u_1}$, where z_{u_1} denotes the first positive zero of u_1.*

Let us first consider the homogeneous linear differential equation

$$u''(t) + a_0 u(t) = 0, \quad a_0 \in \mathbf{R}. \tag{2}$$

The characteristic polynomial of (2), $p(x) = x^2 + a_0$, is an even function, and so the space U_1 of solutions of (2) is invariant under translations and reflections. In particular,

(i) If $p(x) = x^2$, then $U_1 = \langle 1, t \rangle$.
(ii) If $p(x) = x^2 - w^2$, $w \neq 0$, then $U_1 = \langle \cosh(wt), \sinh(wt) \rangle$.
(iii) If $p(x) = x^2 + w^2$, $w \neq 0$, then $U_1 = \langle \cos(wt), \sin(wt) \rangle$.

Let us now consider the second order Cauchy problem

$$\begin{cases} u''(t) + a_0 u(t) = 0, & a_0 \in \mathbf{R}, \\ u(0) = 0, & u'(0) = 1. \end{cases} \tag{3}$$

Let S be the unique solution of (3). In order to start the iterative procedure for obtaining $(n+1)$-dimensional generalized Bernstein bases, let us consider $\alpha < z_S$, where z_S denotes the first positive zero of S, and define two initial functions:

$$u_{0,1}(t) := S(\alpha - t)/S(\alpha), \qquad u_{1,1}(t) := S(t)/S(\alpha), \quad t \in [0, \alpha]. \tag{4}$$

The following result is a consequence of Lemma 1, proves the nonnegativity of the functions defined in (4), and studies their behaviour at the endpoints of $[0, \alpha]$.

Theorem 2. *The space U_1 of solutions of (2) is an extended Chebyshev space on $[0, \alpha]$ if and only if $\alpha < z_S$. Moreover the system $(u_{0,1}, u_{1,1})$ of (4) is a B-basis of U_1 and satisfies*

$$\begin{aligned} &u_{0,1}(0) = u_{1,1}(\alpha) = 1, \qquad u_{1,1}(0) = u_{0,1}(\alpha) = 0, \\ &u'_{0,1}(\alpha) = -1/S(\alpha) < 0, \qquad u'_{1,1}(0) = 1/S(\alpha) > 0. \end{aligned} \tag{5}$$

Proof. If $\alpha \geq z_S$, then U_1 is not an extended Chebyshev space on $[0, \alpha]$ because U_1 is a two-dimensional space and $S \in U_1$ has two zeros: $0, z_S$ on $[0, \alpha]$. Let us now assume that $\alpha < z_S$. The system (S', S) is a canonical basis at 0 of U_1 such that the diagonal entries of $W(S', S)(0)$ are all 1. Thus, from Lemma 1, U_1 is an extended Chebyshev space on $[0, \alpha]$.

Since $S(0) = 0$, $S'(0) = 1$, we have $S(t) > 0$, $\forall t \in (0, \alpha]$ for any $0 < \alpha < z_S$. Hence the functions $u_{0,1}, u_{1,1}$ are nonnegative. Evaluating and differentiating (4), formulae (5) can be immediately obtained. In order to check the linear independence of $u_{0,1}, u_{1,1}$, consider the linear combination $\sum_{i=0}^{1} c_i u_{i,1}(t) = 0$, $t \in [0, \alpha]$. Evaluating at $t = 0$, we get $c_0 = 0$, and evaluating at $t = \alpha$, we get $c_1 = 0$. Therefore the functions $u_{0,1}$ and $u_{1,1}$ are linearly independent solutions of (2), and therefore the system $(u_{0,1}, u_{1,1})$ is a basis of U_1.

Since U_1 is an extended Chebyshev space on $[0, \alpha]$, U_1 has a totally positive basis (u_0, u_1) (see, for instance, Theorem 2.4(iii) of [1]). In addition, this basis (of nonnegative functions) is normalizable (that is, $u_0(t) + u_1(t) \neq 0$ for all $t \in [0, \alpha]$). Otherwise, $\sum_{i=0}^{1} u_i(t) = 0$ for some $t \in [0, \alpha]$, and so $u_i(t) = 0$ for $i = 0, 1$, and all functions of the space would vanish at t. In particular, $u_{i,1}(t) = 0$ for some $t \in [0, \alpha]$ for $i = 0, 1$, contradicting the fact that $(u_{0,1}, u_{1,1})$ is a basis of U_1 formed by nonnegative functions on $[0, \alpha]$ satisfying (5). Besides, from formulae (5) the basis $(u_{0,1}, u_{1,1})$ satisfies

$$\lim_{t \to 0^+} u_{1,1}(t)/u_{0,1}(t) = 0 \quad \text{and} \quad \lim_{t \to \alpha^-} u_{0,1}(t)/u_{1,1}(t) = 0.$$

Then, since U_1 has a normalizable totally positive basis, by the implication (ii) implies (i) of Theorem 3.2 of [17], $(u_{0,0}, u_{1,1})$ is a B-basis. ■

It can be easily checked that

(i) If $p(x) = x^2$, then $S(t) = t$ and $z_S = +\infty$.
(ii) If $p(x) = x^2 - w^2$, $w \neq 0$, then $S(t) = \sinh(wt)/w$ and $z_S = +\infty$.
(iii) If $p(x) = x^2 + w^2$, $w \neq 0$, then $S(t) = \sin(wt)/w$ and $z_S = \pi/w$.

Observe that, except for the case (i), $1 \notin U_1$. This implies that U_1 has no normalized totally positive basis, and therefore it does not possess shape preserving representations. In order to obtain spaces with shape preserving representations including U_1, let us study the space of the integrals of the functions of U_1.

Let us recall that, for a given space of functions $\mathcal{U}$, the space of the derivatives $\mathcal{U}'$ is defined by

$$\mathcal{U}' := \{u' \mid u \in \mathcal{U}\}.$$

The following theorem was obtained in [1] and confirms the equivalence between the existence of normalized B-bases and the existence of extended Chebyshev bases in the space of the derivatives.

Theorem 3. *Let $\mathcal{U}$ be an $(n+1)$-dimensional subspace of $\mathcal{C}^n[a,b]$ such that $1 \in \mathcal{U}$. Then $\mathcal{U}$ is an extended Chebyshev space with a normalized B-basis on $[a,b]$ if and only if the space $\mathcal{U}' := \{u' \mid u \in \mathcal{U}\}$ is extended Chebyshev.*

For $n > 1$, let us now define the $(n+1)$-dimensional spaces U_n such that $U_k' = U_{k-1}$ for all $k = 2, \ldots, n$. We have the following cases:

(i) If $p(x) = x^2$, then

$$U_n = \langle 1, \ldots, t^n \rangle. \tag{6}$$

(ii) If $p(x) = x^2 - w^2$, $w \neq 0$, then

$$U_n = \langle 1, \ldots, t^{n-2}, \cosh(wt), \sinh(wt) \rangle. \tag{7}$$

(iii) If $p(x) = x^2 + w^2$, $w \neq 0$, then

$$U_n = \langle 1, \ldots, t^{n-2}, \cos(wt), \sin(wt) \rangle. \tag{8}$$

Spaces (8) have been considered for $n = 2$ in [12,13,9,8,14], for $n = 3$ in [8] and, in general, in [2,1,11,5]. Spaces (7) have been considered in [6,14]. Here we provide a unified approach which shows simultaneously their extended Chebyshev structure, normalized B-bases on adequate intervals, the justification of their shape preserving properties, and the analysis of their evaluation and subdivision algorithms. Moreover, in Section 5 we consider new spline functions composed from different spaces.

Now, in order to obtain the normalized B-basis of U_n, let us define

$$\begin{aligned}
u_{0,n}(t) &:= 1 - \int_0^t \delta_{0,n-1} u_{0,n-1}(s)\mathrm{d}s, \\
u_{i,n}(t) &:= \int_0^t \left(\delta_{i-1,n-1} u_{i-1,n-1}(s) - \delta_{i,n-1} u_{i,n-1}(s)\right) \mathrm{d}s, \quad i = 1, \ldots, n-1, \\
u_{n,n}(t) &:= \int_0^t \delta_{n-1,n-1} u_{n-1,n-1}(s)\mathrm{d}s,
\end{aligned} \tag{9}$$

for $t \in [0,\alpha]$, where $\delta_{i,n-1} := 1/\int_0^\alpha u_{i,n-1}(s)\,\mathrm{d}s$, $i = 0, \ldots, n-1$.

The following result uses Theorems 2 and 3, and proves that the system $(u_{0,n}, \ldots, u_{n,n})$ defined in (9) is the normalized B-basis of U_n when $\alpha < z_S$.

Theorem 4. *For all $n \geq 2$, U_n is an extended Chebyshev space with a normalized B-basis on $[0,\alpha]$ for any $\alpha < z_S$. The system $(u_{0,n}, \ldots, u_{n,n})$ defined in (9) is the normalized B-basis of U_n. Moreover, at the endpoints of $[0,\alpha]$, it has the same properties as the Bernstein basis of polynomials of degree n on that interval, that is,*

$$\begin{aligned}
&u_{0,n}(0) = u_{n,n}(\alpha) = 1. \\
&u_{i,n}^{(j)}(0) = u_{i,n}^{(k)}(\alpha) = 0, \quad j = 0, \ldots, i-1,\ k = 0, \ldots, n-i-1. \\
&u_{i,n}^{(i)}(0) = \delta_{i-1,n-1}\delta_{i-2,n-2} \cdots \delta_{0,n-i},\ i = 1, \ldots, n. \\
&u_{i,n}^{(n-i)}(\alpha) = (-1)^{n-i}\delta_{i,n-1}\delta_{i,n-2} \cdots \delta_{i,i},\ i = 1, \ldots, n.
\end{aligned} \tag{10}$$

Proof. By Theorem 2 and iterated application of Theorem 3, we can guarantee that U_n is an extended Chebyshev space with a normalized B-basis on $[0,\alpha]$ for any $\alpha < z_S$.

Differentiating in (9), the expression of the derivatives of the functions at the endpoints of $[0,\alpha]$ given in (10) can be obtained by induction on n. Consider a trivial linear combination $\sum_{i=0}^n c_i u_{i,n}(t) = 0$, $t \in [0,\alpha]$. By letting $t = 0$ and taking into account formulae (10), we get $c_0 = 0$. Differentiating and taking again $t = 0$, we deduce that $c_1 = \cdots = c_n = 0$, and that $u_{0,n}, \ldots, u_{n,n}$ are linearly independent. Thus, by construction $(u_{0,n}, \ldots, u_{n,n})$ is a normalized basis of U_n.

Now, let us prove that the system is formed by nonnegative functions on $[0,\alpha]$. As stated before, U_n is an extended Chebyshev space on $[0,\alpha]$. This means that $u_{i,n}$ has at most n zeroes on $[0,\alpha]$. Since, by formulae (10), $t = 0$ is a zero of multiplicity i of $u_{i,n}$ and $t = \alpha$ is a zero of multiplicity $n - i$ of $u_{i,n}$, we deduce that $u_{i,n}(t) \neq 0$, $\forall t \in (0, \alpha)$. Since, by formulae (10), $u_{i,n}^{(i)}(0) > 0$, then $u_{i,n}(t) > 0$, $\forall t \in (0, \alpha)$.

Finally, let us see that the basis defined in (9) is precisely the normalized B-basis of U_n for $n \geq 2$. From (10) we can write

$$\lim_{t \to 0^+} u_{k,n}(t)/u_{j,n}(t) = 0 \quad \text{and} \quad \lim_{t \to \alpha^-} u_{j,n}(t)/u_{k,n}(t) = 0 \tag{11}$$

whenever $j < k$. By Theorem 4, U_n has a normalized B-basis for each $n \geq 2$. Then, since in particular it has a normalized totally positive basis, by the implication (ii) implies (i) of Theorem 3.2 of [17], $(u_{0,n}, \ldots, u_{n,n})$ is the normalized B-basis of U_n. ■

From now on, the system $(u_{0,n}, \ldots, u_{n,n})$ defined in (9), will be called the *generalized Bernstein basis* of U_n.

3. Properties of generalized Bernstein bases

Let us note some interesting properties of the bases (9). First, let us observe that, by (11) and the implication (ii) implies (i) of Theorem 3.1 of [19], they are optimally stable in the sense that there does not exist a basis of nonnegative functions with fewer condition numbers for their evaluation at any point of the domain and for any function of the space (see [19] for the definition of the condition number for the evaluation).

The symmetry of the system $(u_{0,1}, u_{1,1})$ is inherited by the functions of (9), as stated in the following result.

Proposition 5. *The functions of* $(u_{0,n}, \ldots, u_{n,n})$ *satisfy* $u_{n-i,n}(\alpha - t) = u_{i,n}(t)$, $i = 0, \ldots, n$.

Proof. We are going to prove the result by induction. From the Definition (4) of $(u_{0,1}, u_{1,1})$ the result holds for $n = 1$. Assume that the result holds for $n = k$, that is

$$u_{k-i,k}(\alpha - t) = u_{i,k}(t), \quad i = 0, \ldots, k.$$

For $t \in [0, \alpha]$, taking into account that $\delta_{i,k} := 1/\int_0^\alpha u_{i,k}(s)\,\mathrm{d}s$, we can write

$$\int_0^{\alpha - t} u_{k-i,k}(s)\mathrm{d}s = -\int_\alpha^t u_{k-i,k}(\alpha - s)\mathrm{d}s = \int_t^\alpha u_{i,k}(s)\mathrm{d}s = \delta_{i,k}^{-1} - \int_0^t u_{i,k}(s)\mathrm{d}s$$

for $i = 0, \ldots, k$. By letting $t = 0$, we obtain

$$\delta_{i,k} = \delta_{k-i,k}, \quad i = 0, \ldots, k. \tag{12}$$

For any $1 < i < k + 1$, taking into account (9) and (12), we can write

$$\begin{aligned} u_{k-i+1,k+1}(\alpha - t) &= \delta_{k-i,k} \int_0^{\alpha - t} u_{k-i,k}(s)\mathrm{d}s - \delta_{k-i+1,k} \int_0^{\alpha - t} u_{k-i+1,k}(s)\mathrm{d}s \\ &= \delta_{k-i+1,k} \int_0^t u_{k-i+1,k}(\alpha - s)\mathrm{d}s - \delta_{k-i,k} \int_0^t u_{k-i,k}(\alpha - s)\mathrm{d}s \\ &= \delta_{i-1,k} \int_0^t u_{i-1,k}(s)\mathrm{d}s - \delta_{i,k} \int_0^t u_{i,k}(s)\mathrm{d}s = u_{i,k+1}(t). \end{aligned}$$

The proof for the case $i = 0$ and $i = k + 1$ is similar. ■

The following result shows a degree-raising formula.

Proposition 6. *For* $n \geq 2$, *the systems* $(u_{0,n}, \ldots, u_{n,n})$ *of* (9) *satisfy the following degree elevation formula*

$$u_{i,n}(t) = \frac{u_{i,n}^{(i)}(0)}{u_{i,n+1}^{(i)}(0)} u_{i,n+1}(t) + \left(1 - \frac{u_{i+1,n}^{(i+1)}(0)}{u_{i+1,n+1}^{(i+1)}(0)}\right) u_{i+1,n+1}(t), \quad t \in [0, \alpha]. \tag{13}$$

Proof. Clearly, $U_n \subset U_{n+1}$ for all $n \geq 1$. Then we can write $u_{i,n}(t) = \sum_{j=0}^{n+1} c_{j,i} u_{j,n+1}(t)$, $\forall t \in [0, \alpha]$. By formulae (10), we conclude $c_{0,i} = u_{i,n}(0)$ and $c_{n+1,i} = u_{i,n}(\alpha)$. Differentiating k times, we deduce, by formulae (10) again, that $c_{k,i} = 0$, $k = 1, \ldots, i-1$ and $k = i+2, \ldots, n+1$. Therefore

$$u_{i,n}(t) = c_{i,i} u_{i,n+1}(t) + c_{i+1,i} u_{i+1,n+1}(t).$$

Using L'Hospital's Rule, we have

$$c_{i,i} = \frac{u_{i,n}(t) - c_{i+1,i} u_{i+1,n+1}(t)}{u_{i,n+1}(t)} = \lim_{t \to 0^+} \frac{u_{i,n}(t) - c_{i+1,i} u_{i+1,n+1}(t)}{u_{i,n+1}(t)} = \frac{u_{i,n}^{(i)}(0)}{u_{i,n+1}^{(i)}(0)}.$$

Similarly,

$$c_{i+1,i} = \frac{u_{i,n}^{(n-i)}(\alpha)}{u_{i+1,n+1}^{(n-i)}(\alpha)}.$$

We can write

$$1 = \sum_{i=0}^{n} u_{i,n}(t) = \sum_{i=0}^{n} \left(c_{i,i} u_{i,n+1}(t) + c_{i+1,i} u_{i+1,n+1}(t) \right) = \sum_{i=0}^{n+1} u_{i,n+1}(t).$$

From the linear independence of the generalized Bernstein basis functions, we have

$$c_{i+1,i} = \begin{cases} 1 - c_{i+1,i+1}, & i = 0, \ldots, n-1 \\ 1, & i = n+1. \end{cases}$$

Thus we have the degree-elevation formula (13). ∎

The parameter α not only controls the length of the interval of definition. It can also be seen as a shape factor with a tension-like effect in all curves generated by generalized Bernstein bases. The following result will explain what happens when $\alpha \to 0$. In order to prevent generalized Bernstein bases from losing their domain intervals $[0, \alpha]$ for $\alpha \to 0$, we need reparametrization by $\text{new}_\alpha(\tau) := \text{old}_\alpha(\tau\alpha)$ in the following discussions. Then the new functions are defined on fixed intervals [0, 1], and could have the parameter $\alpha \to 0$.

Proposition 7. *For any $n \geq 2$, as $\alpha \to 0$, the normalized B-basis of U_n converges uniformly to the Bernstein basis $(b_{0,n}, \ldots, b_{n,n})$ of degree n on* [0, 1].

Proof. The result can be proved by induction on n. Let $C(t) := \int_0^t S(x)\mathrm{d}x$. It can be checked that

$$u_{0,2}(t) = \frac{C(\alpha - t)}{C(\alpha)}, \qquad u_{1,2}(t) = 1 - \frac{C(\alpha - t)}{C(\alpha)} - \frac{C(t)}{C(\alpha)}, \qquad u_{2,2}(t) = \frac{C(t)}{C(\alpha)},$$

for $t \in [0, \alpha]$. After reparametrizing by $\tau = t/\alpha$, developing by the Taylor expansion at $\tau = 0$ one has

$$u_{2,2}(\tau\alpha) - \tau^2 = \frac{O(\alpha^3)}{\alpha^2/2 + O(\alpha^3)} + \frac{O(\alpha^3)}{\alpha^2 + O(\alpha^3)} \tau^2.$$

Taking into account that

$$\lim_{\alpha \to 0} \frac{O(\alpha^3)}{\alpha^2/2 + O(\alpha^3)} = 0, \qquad \lim_{\alpha \to 0} \frac{O(\alpha^3)}{\alpha^2 + O(\alpha^3)} = 0$$

one immediately deduces that

$$\lim_{\alpha \to 0} \max_{0 \leq \tau \leq 1} |u_{2,2}(\alpha\tau) - \tau^2| = 0.$$

Similarly, it can be proved that $\lim_{\alpha \to 0} \max_{0 \leq \tau \leq 1} |u_{1,2}(\alpha\tau) - 2\tau(1-\tau)| = 0$ and $\lim_{\alpha \to 0} \max_{0 \leq \tau \leq 1} |u_{0,2}(\tau\alpha) - (1-\tau)^2| = 0$.

After reparametrizing by $\tau = t/\alpha$, suppose that, by the inductive hypothesis,

$$v_{i,n-1}(\tau) := u_{i,n-1}(\alpha\tau), \qquad \lim_{\alpha \to 0} v_{i,n-1}(\tau) = b_{i,n-1}(\tau), \quad i = 0, \ldots, n-1.$$

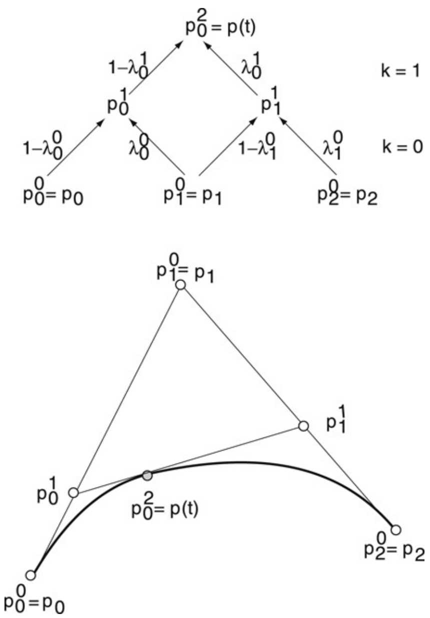

Fig. 1. Subdivision of a generalized Bézier curve in U_2.

By (9), we have

$$u_{i,n}(t) = \frac{\int_0^t u_{i-1,n-1}(s)\mathrm{d}s}{\int_0^\alpha u_{i-1,n-1}(s)\mathrm{d}s} - \frac{\int_0^t u_{i,n-1}(s)\mathrm{d}s}{\int_0^\alpha u_{i,n-1}(s)\mathrm{d}s}.$$

Since $\int_0^1 b_{i,n}(\tau)\mathrm{d}\tau = 1/(n+1)$ we have

$$\begin{aligned}\lim_{\alpha\to 0} v'_{i,n}(\tau) &= \lim_{\alpha\to 0}\left(\frac{v_{i-1,n-1}(\tau)}{\int_0^1 u_{i-1,n-1}(\alpha\tau)\mathrm{d}\tau} - \frac{v_{i,n-1}(\tau)}{\int_0^1 u_{i,n-1}(\alpha\tau)\mathrm{d}\tau}\right)\\ &= n\left(b_{i-1,n-1}(\tau) - b_{i,n-1}(\tau)\right) = b'_{i,n}(\tau).\end{aligned}$$

Finally, since $u_{i,n}(0) = b_{i,n}(0)$ we have $\lim_{\alpha\to 0} v_{i,n}(\tau) = b_{i,n}(\tau)$. ∎

Given the basis $(u_{0,n}, \ldots, u_{0,n})$ of (9) defined on an interval $I = [0,\alpha]$, $0 < \alpha < z_S$, and a control polygon $P_0 \cdots P_n$, a *generalized Bézier curve* in $U_n\gamma(t)$ is the parametric curve $\gamma(t) = \sum_{i=0}^n P_i u_{i,n}(t)$, $t \in I = [0,\alpha]$. Since $(u_{0,n}, \ldots, u_{0,n})$ is the normalized B-basis of U_n, it has optimal shape preserving properties according to [18,17], and γ preserves the shape properties of its control polygon [15,17].

Keeping the same control polygon, as the parameter α varies we are not simply changing the domain of a single curve, but defining different curves. From Proposition 7, when $\alpha = 0$ the generalized Bézier curve $\gamma = \sum_{i=0}^n P_i u_{i,n}(t)$ transforms to an integral Bézier curve with control points P_i.

4. Subdivision and evaluation algorithms

As generalized Bézier curves are expressed in terms of a normalized B-basis, they admit a corner cutting algorithm, called B-algorithm (see [20]), that provides for their evaluation and subdivision.

Let us first consider a generalized Bézier curve in U_2

$$\gamma(s) = P_0 u_{0,2}(s) + P_1 u_{1,2}(s) + P_2 u_{2,2}(s), \quad s \in [0,\alpha], \qquad \alpha < z_S,$$

and a parameter value $t \in [0,\alpha]$. There exist certain values $\lambda_i^k(t)$ that define the intermediate points $P_i^k(t)$ of a de Casteljau-like algorithm (Fig. 1):

$$P_i^{k+1}(t) = (1-\lambda_i^k(t))P_i^k(t) + \lambda_i^k(t)P_{i+1}^k(t), \qquad \begin{array}{l} k = 0,1 \\ i = 0, 1-k. \end{array} \tag{14}$$

Beginning with $P_i^0 := P_i$, $i = 0, 1, 2$, this algorithm yields the final value $P_0^2 = \gamma(t)$. In addition, the two segments in which the parameter t divides the curve have control points given by $\{P_0^0, P_0^1, P_0^2\}$ and $\{P_0^2, P_1^2, P_2^0\}$, respectively. We are going to derive compact expressions of $\lambda_i^k(t)$ for an arbitrary $t \in [0, \alpha]$. The trick is to subdivide the function $p(t) = S(t)$, obtain the solution of the initial value problem (3), contained in U_2, and isolate $\lambda_i^k(t)$ from the barycentric combinations (14) in this particular case.

By defining $C(t) := \int_0^t S(x)\mathrm{d}x$, the normalized B-bases of U_2 on the intervals $[0, t]$ and $[t, \alpha]$ can be written as follows

$$\left(\frac{C(t-u)}{C(t)}, 1 - \frac{C(t-u)}{C(t)} - \frac{C(u)}{C(t)}, \frac{C(u)}{C(t)}\right), \quad u \in [0, t], \tag{15}$$

$$\left(\frac{C(\alpha-u)}{C(\alpha-t)}, 1 - \frac{C(\alpha-u)}{C(\alpha-t)} - \frac{C(u-t)}{C(\alpha-t)}, \frac{C(u-t)}{C(\alpha-t)}\right), \quad u \in [t, \alpha], \tag{16}$$

respectively.

It can be easily checked that, in the normalized B-basis of U_2 on $[0, \alpha]$, the function $S(t)$, $t \in [0, \alpha]$ has the coefficients:

$$\{0, C(\alpha)/S(\alpha), S(\alpha)\}. \tag{17}$$

Similarly, it can be checked that the coefficients of $S(u)$, $u \in [0, t]$ with respect to the normalized B-basis on $[0, t]$ given in (15) are

$$\{0, C(t)/S(t), S(t)\}, \tag{18}$$

and the coefficients of $S(u)$, $u \in [t, \alpha]$ with respect to the normalized B-basis on $[t, \alpha]$ given in (16) are

$$\left\{S(t), S(\alpha) - S'(\alpha)\frac{C(\alpha-t)}{S(\alpha-t)}, S(\alpha)\right\}. \tag{19}$$

Subdividing at an arbitrary value t, isolation of the coefficients $\lambda_i^k(t)$ yields:

$$\lambda_0^0(t) = \frac{S(\alpha)}{C(\alpha)}\frac{C(t)}{S(t)}, \qquad \lambda_1^0(t) = 1 - \frac{S'(\alpha)S(\alpha)}{S(\alpha)^2 - C(\alpha)}\frac{C(\alpha-t)}{S(\alpha-t)},$$

$$\lambda_0^1(t) = \frac{(S(t)^2 - C(t))S(\alpha-t)}{S(\alpha)S(t)S(\alpha-t) - S'(\alpha)C(\alpha-t)S(t) - C(t)S(\alpha-t)}.$$

It can be checked that, if $U_2 = \langle 1, t, t^2\rangle$, $t \in [0, \alpha]$, then

$$\lambda_0^0(t) = t/\alpha, \qquad \lambda_1^0(t) = t/\alpha, \qquad \lambda_0^1(t) = t/\alpha.$$

If $U_2 = \langle 1, \cos t, \sin t\rangle$, $t \in [0, \alpha]$, then

$$\lambda_0^0(t) = \frac{\sin\alpha(1-\cos t)}{\sin t(1-\cos\alpha)}, \qquad \lambda_1^0(t) = \frac{\sin t + \sin(\alpha-t) - \sin\alpha}{(1-\cos\alpha)\sin(\alpha-t)},$$

$$\lambda_0^1(t) = \frac{(1-\cos t)\sin(\alpha-t)}{\sin t + \sin(\alpha-t) - \sin\alpha}.$$

If $U_2 = \langle 1, \cosh t, \sinh t\rangle$, $t \in [0, \alpha]$, then

$$\lambda_0^0(t) = \frac{\sinh\alpha(\cosh t - 1)}{\sinh t(\cosh\alpha - 1)}, \qquad \lambda_1^0(t) = \frac{\sinh\alpha - \sinh t - \sinh(\alpha-t)}{(\cosh\alpha - 1)\sinh(\alpha-t)},$$

$$\lambda_0^1(t) = \frac{(\cosh t - 1)\sinh(\alpha-t)}{\sinh\alpha - \sinh t - \sinh(\alpha-t)}.$$

By defining the matrices

$$\Lambda_1(t) := \begin{pmatrix} 1-\lambda_0^1(t) & \lambda_0^1(t)\end{pmatrix}, \qquad \Lambda_2(t) := \begin{pmatrix} 1-\lambda_0^0(t) & \lambda_0^0(t) & 0 \\ 0 & 1-\lambda_1^0(t) & \lambda_1^0(t)\end{pmatrix} \tag{20}$$

we can write

$$(u_{0,2}(t), u_{1,2}(t), u_{1,2}(t)) = \Lambda_1(t)\Lambda_2(t), \quad \forall t \in [0, \alpha]. \tag{21}$$

This factorization of the normalized B-basis of U_2 describes the B-algorithm for the evaluation and subdivision of generalized Bézier curves in U_2.

It is easy to check that $\lambda^i_j(t)$, $i = 0, 1$, $j = 0, 1 - i$, is a strictly increasing function on $[0, \alpha]$ such that $\lambda^i_j(0) = 0$ and $\lambda^i_j(\alpha) = 1$. Clearly, since for $\alpha \to 0$ the generalized Bézier curve degenerates to a Bézier curve with control points P_i, the coefficients λ^k_i degenerate to t/α. In other words, the B-algorithm reduces to the standard de Casteljau algorithm. This property can be easily checked by introducing the Taylor expansions of $\lambda^i_j(t)$, $i = 0, 1$, $j = 0, 1 - i$, and taking limits.

Let us recall that the de Casteljau algorithm for the pointwise evaluation of a Bézier curve is based on the following well-known recurrence relations of the Bernstein basis on [0, 1]:

$$b_{i,n+1}(t) = \lambda_{i-1}(t)b_{i-1,n}(t) + (1 - \lambda_i(t))b_{i,n}(t), \quad i = 0, \ldots, n + 1, \tag{22}$$

where $\lambda_i(t) := t$ for $i = 0, \ldots, n$, $\lambda_{-1}(t) := 0$, and $\lambda_{n+1}(t) := 1$. Indeed, we can write (22) as

$$(b_{0,n+1}(t), \ldots, b_{n+1,n+1}(t)) = (b_{0,n}(t), \ldots, b_{n,n}(t))\Lambda(t), \tag{23}$$

where $\Lambda(t)$ denotes the nonnegative stochastic bidiagonal matrix

$$\Lambda(t) = \begin{pmatrix} 1 - \lambda_0(t) & \lambda_0(t) & & \\ & \ddots & \ddots & \\ & & 1 - \lambda_n(t) & \lambda_n(t) \end{pmatrix}. \tag{24}$$

Then, starting with a Bézier curve $\gamma(t) = \sum_{i=0}^{n+1} Q_i b_{i,n+1}(t)$, equality (23) gives

$$\gamma(t) = (b_{0,n+1}, \ldots, b_{n+1,n+1})(Q_0, \ldots, Q_{n+1})^{\mathrm{T}} = (b_{0,n}, \ldots, b_{n,n})(P_0(t), \ldots, P_n(t))^{\mathrm{T}}$$

where

$$(P_0(t), \ldots, P_n(t))^{\mathrm{T}} := \Lambda(t)(Q_0, \ldots, Q_{n+1})^{\mathrm{T}}. \tag{25}$$

Equality (25) describes the first step of the de Casteljau algorithm for the evaluation of $\gamma(t)$.

By means of the degree raising technique, we can express the generalized Bézier curve in the space $U_n \gamma(t) = \sum_{i=0}^{n} P_i u_{i,n}(t)$, $t \in [0, \alpha]$, in terms of the generalized Bézier basis of one higher dimension: $\gamma(t) = \sum_{i=0}^{n+1} Q_i u_{i,n+1}(t)$, $t \in [0, \alpha]$. Indeed, the relations (13) can be written in matrix form as

$$(u_{0,n}, \ldots, u_{n,n}) = (u_{0,n+1}, \ldots, u_{n+1,n+1})B_n, \tag{26}$$

where B_n is an $(n + 2) \times (n + 1)$ nonnegative stochastic bidiagonal matrix. Such a matrix can be written as:

$$B_n = \begin{pmatrix} 1 & 0 & \cdots & \cdots & 0 \\ \beta_1 & 1 - \beta_1 & 0 & \cdots & 0 \\ 0 & \beta_2 & 1 - \beta_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \beta_n & 1 - \beta_n \\ 0 & \cdots & \cdots & 0 & 1 \end{pmatrix}, \quad 0 \le \beta_i \le 1, \tag{27}$$

$1 \le i \le n$. Equality (26) corresponds to the choice

$$\beta_i := 1 - \frac{u^{(i)}_{i,n}(0)}{u^{(i)}_{i,n+1}(0)}, \quad i = 1, \ldots, n. \tag{28}$$

Using (26), we can write:

$$\gamma(t) = (u_{0,n}, \ldots, u_{n,n})(P_0, \ldots, P_n)^{\mathrm{T}} = (u_{0,n+1}, \ldots, u_{n+1,n+1})B_n(P_0, \ldots, P_n)^{\mathrm{T}},$$

which proves that the new control polygon is given by

$$(Q_0, \ldots, Q_{n+1})^{\mathrm{T}} := B_n(P_0, \ldots, P_n)^{\mathrm{T}}. \tag{29}$$

In [21], the existence of nondecreasing functions $\lambda_0^n(t), \ldots, \lambda_n^n(t)$, $t \in [0, \alpha]$ with values in [0, 1] such that

$$(u_{0,n+1}(t), \ldots, u_{n+1,n+1}(t)) = (u_{0,n}(t), \ldots, u_{n,n}(t))\Lambda_n(t), \quad t \in I. \tag{30}$$

were proved. The matrix $(n+1) \times (n+2)\Lambda_n(t)$ is defined from $\lambda_i^n(t)$ as

$$\Lambda_n(t) = \begin{pmatrix} 1-\lambda_0^n(t) & \lambda_0^n(t) & & \\ & \ddots & \ddots & \\ & & 1-\lambda_n^n(t) & \lambda_n^n(t) \end{pmatrix}. \tag{31}$$

The following proposition of [21] was devoted to showing that formula (30) holds for normalized totally positive bases.

Proposition 8. *Let $(u_{0,n+1}, \ldots, u_{n+1,n+1})$, $(u_{0,n}, \ldots, u_{n,n})$ be two normalized totally positive bases of functions on I related by (26), where B_n is a matrix (27) (rank$B_n = n+1$). Let $C_i^n := \{t \in I \mid u_{i,n}(t) \neq 0\}$. Then the functions $\lambda_i : I \to \mathbf{R}$, $i = 0, \ldots, n$, defined by*

$$\lambda_i^n(t) := \begin{cases} \beta_{i+1} \inf\left\{ \left. \dfrac{u_{i+1,n+1}(s)}{u_{i,n}(s)} \right| s \in C_i^n \right\}, & \text{if } u_{i,n}(s) = 0, \forall s \le t, \\ \beta_{i+1} \sup\left\{ \left. \dfrac{u_{i+1,n+1}(s)}{u_{i,n}(s)} \right| s \in C_i^n, s \le t \right\}, & \text{otherwise}, \end{cases} \tag{32}$$

are nondecreasing, and satisfy

$$0 \le \lambda_i^n(t) \le 1, \quad \forall t \in I,\ i = 0, \ldots, n. \tag{33}$$

Furthermore, if we use Definition (31), then (30) holds.

When dealing with normalized B-bases, the previous result can be improved as follows.

Proposition 9. *Let $(u_{0,n+1}, \ldots, u_{n+1,n+1})$ be the normalized B-basis of a vector space $\mathcal{U}^{n+1}$ of functions defined on $I \subseteq \mathbf{R}$, B_n a matrix (27) (rank $B_n = n+1$), and let*

$$(u_{0,n}, \ldots, u_{n,n}) := (u_{0,n+1}, \ldots, u_{n+1,n+1})B_n.$$

The functions $\lambda_i^n : I \to \mathbf{R}$ ($i = 0, \ldots, n$) defined in (32) are increasing and verify:

(a) *if $\beta_{i+1} = 0$, then $\lambda_i^n(t) = 0$, $\forall t \in I$,*
(b) *if $\beta_{i+1} = 1$, then $\lambda_i^n(t) = 1$, $\forall t \in I$,*
(c) *if $0 < \beta_{i+1} < 1$, then λ_i^n is increasing in I, $0 \le \lambda_i^n(t) \le 1$, $\forall t \in I$, $\inf\{\lambda_i^n(t) \mid t \in I\} = 0$, $\sup\{\lambda_i^n(t) \mid t \in I\} = 1$.*

Proof. (a) If $\beta_{i+1} = 0$, then $u_{i,n}(t) = u_{i,n+1}(t)$ and by (32) $\lambda_i^n(t) = 0$, $\forall t \in I$.
(b) If $\beta_{i+1} = 1$, then $u_{i,n}(t) = u_{i+1,n+1}(t)$ and by (32) $\lambda_i^n(t) = 1$, $\forall t \in I$.
(c) Let us assume that $0 < \beta_{i+1} < 1$. By Proposition 8, λ_i^n is monotonic increasing and $0 \le \lambda_i^n(t) \le 1$, $\forall t \in I$. Since $(u_{0,n+1}, \ldots, u_{n+1,n+1})$ is a normalized B-basis, $u_{i,n+1}/u_{i+1,n+1}$ is decreasing in $C_{i+1}^{n+1} := \{t \in I | u_{i+1,n+1}(t) \neq 0\}$ and satisfies $\inf\{u_{i,n+1}(t)/u_{i+1,n+1}(t) | t \in C_{i+1}^{n+1}\} = 0$. Taking into account that $\lambda_i^n(t) \le 1$, $\forall t \in I$ and (32), we can write

$$1 \ge \sup\left\{\lambda_i^n(t) | t \in I\right\} \ge \sup\left\{ \left. \beta_{i+1} \frac{u_{i+1,n+1}(t)}{u_{i,n}(t)} \right| t \in C_i^n \right\}. \tag{34}$$

Since $\beta_{i+1} > 0$ and letting $\beta_n := 1$, we can write

$$\beta_{i+1}u_{i+1,n+1}(t) = \lambda_i^n(t)u_{i,n}(t), \quad \forall t \in I,\ i = 0,\ldots,n. \tag{35}$$

Then $C_i^n \supseteq C_{i+1}^{n+1}$, and so

$$\begin{aligned}\sup\left\{\beta_{i+1}\frac{u_{i+1,n+1}(t)}{u_{i,n}(t)}\,\middle|\, t\in C_i^n\right\} &\geq \sup\left\{\beta_{i+1}\frac{u_{i+1,n+1}(t)}{u_{i,n}(t)}\,\middle|\, t\in C_{i+1}^{n+1}\right\}\\ &= \sup\left\{\beta_{i+1}\frac{u_{i+1,n+1}(t)}{(1-\beta_i)u_{i,n+1}(t)+\beta_{i+1}u_{i+1,n+1}(t)}\,\middle|\, t\in C_{i+1}^{n+1}\right\}\\ &= \frac{\beta_{i+1}}{(1-\beta_i)\inf\{u_{i,n+1}(t)/u_{i+1,n+1}(t)|t\in C_{i+1}^{n+1}\}+\beta_{i+1}} = 1.\end{aligned} \tag{36}$$

By (34) and (36), we deduce that $1 = \sup\{\lambda_i^n(t)|t \in I\}$. Moreover, $u_{i+1,n+1}/u_{i,n+1}$ is increasing in $C_i^{n+1} := \{t \in I|u_{i,n+1}(t) \neq 0\}$ and $\inf\{u_{i+1,n+1}(t)/u_{i,n+1}(t)|t \in C_i^{n+1}\} = 0$. Taking into account the fact that $0 \leq \lambda_i(t)$ and that $\lambda_i^n(t) \leq \frac{\beta_{i+1}u_{i+1,n+1}(t)}{(1-\beta_i)u_{i,n+1}(t)}$, $\forall t \in C_i^{n+1}$, we deduce

$$0 \leq \inf\{\lambda_i^n(t)|t\in C_i^{n+1}\} \leq \inf\left\{\frac{\beta_{i+1}u_{i+1,n+1}(t)}{(1-\beta_i)u_{i,n+1}(t)}\,\middle|\, t\in C_i^{n+1}\right\} = 0. \quad \blacksquare$$

Remark 10. Observe that in our case, the normalized B-bases of U_n and U_{n+1}, $(u_{0,n},\ldots,u_{n,n})$ and $(u_{0,n+1},\ldots,u_{n+1,n+1})$, are formed by continuous functions, and then we can write

$$\lambda_i^n(t) = \beta_{i+1}\frac{u_{i+1,n+1}(t)}{u_{i,n}(t)}, \quad t\in C_i^n$$

and deduce the continuity of $\lambda_i^n(t)$. Let us also notice that, taking into account Proposition 9, the functions $\lambda_i^n(t)$, $i = 0,\ldots,n$, can be easily computed from the normalized B-basis $(u_{0,n},\ldots,u_{n,n})$ of U_n at the same time as $(u_{0,n+1},\ldots,u_{n+1,n+1})$, the normalized B-basis of U_{n+1}, without calculating the coefficients β_i. By Proposition 9, $\lim_{t\to\alpha}\lambda_i^n(t) = 1$, and therefore we can write

$$\lambda_i^n(t) = \lim_{s\to\alpha}\left(\frac{u_i^n(s)}{u_{i+1}^{n+1}(s)}\right)\frac{u_{i+1}^{n+1}(t)}{u_i^n(t)}. \tag{37}$$

Example 1. Let us illustrate Remark 10 by considering the space

$$H_3 := \langle 1, t, \cosh t, \sinh t\rangle, \quad t\in[0,\alpha],$$

and obtaining the corner cutting algorithm for its evaluation at the same time as the computation of its normalized B-basis. We shall start with the normalized B-basis $(u_{0,2}, u_{1,2}, u_{2,2},)$ of $H_2 := \langle 1, \cosh t, \sinh t\rangle$. Using (9) it can be checked that

$$u_{0,2}(t) = \frac{\cosh(\alpha-t)-1}{\cosh\alpha-1}, \qquad u_{1,2}(t) = 1-u_{0,2}(t)-u_{2,2}(t), \qquad u_{2,2}(t) = \frac{\cosh t-1}{\cosh\alpha-1}$$

and $\delta_{i,2} = 1/\int_0^\alpha u_{i,2}(s)\mathrm{d}s$ $i = 0, 1, 2$ are

$$\delta_{0,2} = \frac{\cosh\alpha-1}{\sinh\alpha-\alpha}, \qquad \delta_{1,2} = \frac{\cosh\alpha-1}{\alpha(1+\cosh\alpha)-2\sinh\alpha}, \qquad \delta_{2,2} = \frac{\cosh\alpha-1}{\sinh\alpha-\alpha}.$$

Using (9), we compute the normalized B-basis of H_3, obtaining

$$u_{3,3}(t) = \int_0^t \delta_{2,2}u_{2,2}(s)\mathrm{d}s = \frac{t-\sinh t}{\alpha-\sinh\alpha}.$$

Now, computing $\lim_{t\to\alpha} u_{3,3}(t)/u_{2,2}(t)$ and using (37) we obtain

$$\lambda_2^2(t) = \frac{1-\cosh\alpha}{\alpha-\sinh\alpha}\,\frac{t-\sinh t}{1-\cosh t}.$$

Using (9) again,

$$u_{2,3}(t) = \frac{\sinh\alpha(1-\cosh\alpha)(t-\sinh t)}{(\alpha(1+\cosh\alpha)-2\sinh\alpha)(\alpha-\sinh\alpha)} - \frac{\sinh\alpha(1-\cosh t)}{\alpha(1+\cosh\alpha)-2\sinh\alpha}.$$

Then, computing $\lim_{t\to\alpha} u_{2,3}(t)/u_{1,2}(t)$ and using (37), we obtain

$$\lambda_1^2(t) = \frac{\sinh\alpha\left((1-\cosh\alpha)(t-\sinh t)+(\sinh\alpha-\alpha)(\cosh t-1)\right)}{(2-2\cosh\alpha+\alpha\sinh\alpha)\left((1+\cosh\alpha)(1-\cosh t)+\sinh\alpha\sinh t\right)}.$$

Finally, since $u_{1,3}(t) = u_{2,3}(\alpha-t)$ and $u_{0,3}(t) = u_{0,3}(\alpha-t)$, it can be deduced that

$$\lambda_0^2(t) = 1-\lambda_2^2(\alpha-t).$$

Let us observe that the spaces U_n $n \geq 2$ form a chain, that is,

$$U_n \supset U_{n-1} \supset \cdots \supset U_2.$$

Moreover, the corresponding generalized Bernstein bases $(u_{0,k},\ldots,u_{k,k})$ satisfy

$$(u_{0,k}(t),\ldots,u_{k,k}(t)) := (u_{0,k+1}(t),\ldots,u_{k,k+1}(t))B_{k+1}, \quad k = n-1,\ldots,2 \tag{38}$$

where $B_{k+1} \in \mathbf{R}^{(k+2)\times(k+1)}$ is a matrix of type (27), and rank $B_{k+1} = k+1$.

By Proposition 8, the generalized Bernstein bases are related by

$$(u_{0,k+1}(t),\ldots,u_{k+1,k+1}(t)) = (u_{0,k}(t),\ldots,u_{k,k}(t))\Lambda_{k+1}(t), \quad t \in [0,\alpha], \tag{39}$$

for $k \geq 2$, where $\Lambda_{k+1}(t)$ is a $(k+1)\times(k+2)$ matrix of type (31). We shall denote by $\lambda_i^{k+1}(t)$ the $(i+1, i+2)$ entry of $\Lambda_{k+1}(t)$. The recurrences (38) and (39) give

$$(u_{0,k}(t),\ldots,u_{k,k}(t)) = (u_{0,n}(t),\ldots,u_{n,n}(t))B_n\cdots B_{k+2}B_{k+1}, \quad t \in I \tag{40}$$

$$(u_{0,k}(t),\ldots,u_{k,k}(t)) = (u_{0,2}(t),u_{1,2}(t),u_{2,2}(t))\Lambda_3(t)\cdots\Lambda_k(t), \quad t \in I \tag{41}$$

for $k \geq 2$. Taking into account now the factorization (21) of the generalized Bernstein basis of U_2, for any control polygon $P_0\cdots P_n$, the following generalization of the de Casteljau algorithm

for $j = 0,1,\ldots,n$
  $P_j^0(t) := P_j$
for $i = 1,\ldots,n$
  for $j = 0,1,\ldots,i$
    $P_j^i(t) := (1-\lambda_j^{i+1}(t))P_j^{i-1}(t) + \lambda_j^{i+1}(t)P_{j+1}^{i-1}(t)$

satisfies $\gamma(t) = P_0^0(t)$ for all $t \in I$; that is, this generalized de Casteljau algorithm reconstructs the curve from its control polygon.

As stated before, the derivative $\gamma'(t)$ of a generalized Bézier curve in U_n is, clearly, a generalized Bézier curve in U_{n-1}

$$\gamma'(t) = \sum_{i=0}^{n-1} Q_i u_{i,n-1}(t), \quad t \in [0,\alpha],$$

where $Q_i = \delta_{i,n-1}(P_{i+1} - P_i),\ i = 0, \ldots, n-1$. That means that the algorithm

$$\begin{aligned}
&\textbf{for } j = 0, 1, \ldots, n-1\\
&\quad Q_j^0(t) := \delta_{i,n-1}\,(P_{i+1} - P_i)\\
&\textbf{for } i = 1, \ldots, n-1\\
&\quad \textbf{for } j = 0, 1, \ldots, i\\
&\qquad Q_j^i(t) := (1 - \lambda_j^{i+1}(t))Q_j^{i-1}(t) + \lambda_j^{i+1}(t)Q_{j+1}^{i-1}(t)
\end{aligned}$$

satisfies $\gamma'(t) = Q_0^0(t)$ for all $t \in I$.

5. An integral construction of a B-spline basis through one or several spaces

A real object often has a complex geometry and piecewise functions, composed by different spaces (6) and (7) or (8), would be very useful. Given $[a, b]$, let us consider the knot sequence τ

$$\tau = \{a = t_{-1} = t_0 \le t_1 \le \cdots \le t_{n-1} \le t_n = t_{n+1} = b\}.$$

Let $I_i := [t_i, t_{i+1}]$ and $h_i := t_{i+1} - t_i$ for $i = -1, \ldots, n$. We denote the collection of generalized B-splines of order k defined on τ by $S_{k,\tau}$. On each interval I_i, every function of $S_{k,\tau}$ is a function in the space $U_k^{(i)}$ of (6) and (7) or (8), and at t_i it is $k - m_i$ times continuously differentiable where $m_i \ge 1$ is the number of times t_i appears in $\{t_{i-k}, \ldots, t_{i+1}\}$. It can be checked that the usual operations of addition and scalar multiplication of functions of $S_{k,\tau}$ are closed, i.e., $S_{k,\tau}$ is a linear space. The proof of the following result is straightforward.

Lemma 11. *If, for some $i \in \{0, \ldots, n-1\}$, $1 \notin U_1^{(i)}$, then there exists no normalized basis of $S_{1,\tau}$.*

In order to construct a basis of $S_{k,\tau}$ $k \ge 2$, let us use the convention $0/0 := 0$ and define the following set of piecewise functions on the spaces U_1 of (6)–(8):

$$N_{i,1}(t) := \begin{cases} S_{i-1}\,(t - t_{i-1})\,/S_{i-1}\,(h_{i-1})\,, & t \in [t_{i-1}, t_i],\\ S_i\,(t_{i+1} - t)\,/S_i\,(h_i)\,, & t \in (t_i, t_{i+1}],\\ 0, & \text{elsewhere,} \end{cases} \tag{42}$$

for $i = 0, \ldots, n$, where S_{i-1} and S_i are any of the possible solutions of the second order Cauchy problem (3); that is, $S_j(t) = t$ or $S_j(t) = \sinh(wt)/w$ or $S_j(t) = \sin(wt)/w$ for $j = 0, \ldots, n-1$. We shall call the system $(N_{0,1}, \ldots, N_{n,1})$ given in (42) the *generalized B-spline basis of order* 1.

Lemma 12. *If $h_i < z_{S_i}$ for all $i = 0, \ldots, n-1$, the system $(N_{0,1}, \ldots, N_{n,1})$ given in (42) is formed by nonnegative functions. Every function $N_{i,1}$ $i = 0, \ldots, n$ vanishes at t_{i-1} and t_{i+1}. Moreover $N_{i,1}$ is $1 - m_i$ continuous at t_i where $m_i \ge 1$ is the number of times t_i appears in $\{t_{i-1}, t_i, t_{i+1}\}$.*

Proof. Let us recall that, since S_i is the solution of (3), $S_i(t) > 0$ for all $0 < t < z_{S_i}$. Therefore, if $h_i < z_{S_i}$, then $N_{i,1}$ is nonnegative for all i. Clearly, by construction, supp $N_{i,1} = [t_{i-1}, t_{i+1}]$. If $t_{i-1} < t_i < t_{i+1}$, then $N_{i,1}$ is a continuous function on $\mathbf{R}$; if $t_{i-1} = t_i < t_{i+1}$, then $S_{i-1}(t - t_{i-1})/S_{i-1}(h_{i-1}) = 0/0 := 0$, and so $N_{i,1}$ is not continuous at $t_{i-1} = t_i$; and, if $t_{i-1} < t_i = t_{i+1}$, then $S_i(t_{i+1} - t)/S_i(h_i) = 0/0 := 0$, and so $N_{i,1}$ is not continuous at $t_i = t_{i+1}$. Finally, if $t_{i-1} = t_i = t_{i+1}$ then $N_{i,1} \equiv 0$. ■

Let us notice that for the particular knot vector $\tau = \{t_{-1} = t_0 < t_1 = t_2\}$, the system (42) coincides with the system given in (4), which is a B-basis of $U_1 = S_{1,\tau}$ if and only if $t_1 - t_0 < z_{S_0}$ (see Theorem 2).

Theorem 13. *If $0 \le h_i < z_{S_i'}$ for all $i = 0, \ldots, n-1$, then the nonzero functions of the system $(N_{0,1}, \ldots, N_{n,1})$ given in (42) form a B-basis of $S_{1,\tau}$.*

Proof. By construction, supp $N_{i,1} = [t_{i-1}, t_{i+1}]$, and so, for all $j \notin \{i, i+1\}$, $N_{j,1}(t) = 0\,\forall t \in I_i$. Suppose $t_i < t_{i+1}$. Substituting in any trivial linear combination of the nonzero functions of $(N_{0,1}, \ldots, N_{n,1})$ with coefficients c_i, we have $\sum_{j=i}^{i+1} c_i N_{i,1}(t) = 0, \forall t \in I_i$. Moreover $N_{i,1}(t) = S_i(t_{i+1} - t)/S_i(h_i)$ and $N_{i+1,1}(t) = S_i(t - t_i)/S_i(h_i)$. Letting $t = t_{i+1}$, since $N_{i,1}(t_{i+1}) = 0$ we deduce $c_{i+1} = 0$. Then, we have $c_i N_{i,1}(t) = 0\,\forall t \in I_i$ and so $c_i = 0$. Since $U_1^{(i)}$ is a two dimensional space, the restrictions to I_i of $N_{i,1}$ and $N_{i+1,1}$ form a basis of $U_1^{(i)}$. By considering all I_j such that $t_j < t_{j+1}$ we obtain, with the same reasoning, that the nonzero functions of $(N_{0,1}, \dots, N_{n,1})$ form a basis of $S_{1,\tau}$.

On the other hand, if $0 < t_{i+1} - t_i < z_{S_i'}$, then $N_{i,1}$ is strictly decreasing on I_i, $N_{i+1,1}$ is strictly increasing on I_i, and so $N_{i,1}/N_{i+1,1}$ is strictly decreasing on I_i. Therefore, for $\tau_1 < \tau_2$ on I_i we have

$$\det M \begin{pmatrix} N_{i,1}(\tau_1) & N_{i+1,1}(\tau_1) \\ N_{i,1}(\tau_2) & N_{i+1,1}(\tau_2) \end{pmatrix} > 0. \tag{43}$$

Taking into account (43), the total positivity of all collocation matrices of $(N_{0,1}, \dots, N_{n,1})$ can be easily deduced. It can be also checked that

$$\inf\left\{N_{i,1}(t)/N_{j,1}(t) \mid N_{j,1}(t) \neq 0\right\} = 0, \qquad \inf\left\{N_{j,1}(t)/N_{i,1}(t) \mid N_{i,1}(t) \neq 0\right\} = 0$$

whenever $j < i$. Then the nonzero functions of $(N_{0,1}, \dots, N_{n,1})$ form a B-basis of $S_{1,\tau}$. ■

For any integer $k > 1$, let us now define the knot vector

$$\tau_k := \{a = t_{-k} = \cdots = t_{-1} = t_0 < t_1 \leq \cdots \leq t_{n-1} < t_n = t_{n+1} = \cdots = t_{n+k} = b\}.$$

From Lemma 11, if for some $i \in \{0, \dots, n-1\}$, $1 \notin U_1^{(i)}$, there exists no generalized B-spline basis of $S_{1,\tau}$. In order to obtain the generalized B-spline basis of S_{k,τ_k} $k > 1$ let us define:

$$\begin{aligned} N_{0,k} &:= 1 - \int_{-\infty}^{t} \delta_{0,k-1} N_{0,k-1}(s)\mathrm{d}s, \\ N_{i,k} &:= \int_{-\infty}^{t} \left(\delta_{i-1,k-1} N_{i-1,k-1}(s) - \delta_{i,k-1} N_{i,k-1}(s)\right) \mathrm{d}s, \quad i = 1, \dots, n+k-2, \\ N_{n+k-1,k} &:= \int_{-\infty}^{t} \delta_{n+k-2,k-1} N_{n+k-2,k-1}(s)\mathrm{d}s, \end{aligned} \tag{44}$$

where $\delta_{i,k-1} := 1/\int_{-\infty}^{\infty} N_{i,k-1}(s)\,\mathrm{d}s$ for $i = 0, \dots, n+k-2$ and $k > 1$. In order to ensure that the functions $N_{i,k}$ form a normalized system when some $N_{i,k-1} \equiv 0$ (this happens if $t_{i-k+1} = \cdots = t_{i+1}$), we set $\delta_{i,k-1} N_{i,k-1} := 0$ and

$$\int_{-\infty}^{t} \delta_{i,k-1} N_{i,k-1}(s)\mathrm{d}s := \begin{cases} 0, & t < t_{i-k+1}, \\ 1, & t \geq t_{i-k+1}. \end{cases}$$

We shall call the system $(N_{0,k}, \dots, N_{n+k-1,k})$ given in (44) the *generalized B-spline basis of order* k.

Fig. 2 illustrates the generalized B-spline bases of order 1 and 2 obtained by (42) and (44), respectively, for the knot vector $\tau = \{0, 0, \pi/4, 3, 3\}$ by considering $S_0(t) = \sin t$ and $S_1(t) = \sinh t$.

Some basic properties of the systems $(N_{0,k}, \dots, N_{n+k-1,k})$, $k > 1$ can be easily deduced from their definition (44), using an inductive argument.

(1) *Local support.* By definition (44), $\operatorname{supp} N_{i,k} = [t_{i-k}, t_{i+1}]$ for $i = 0, \dots, n+k-1$ and $k \geq 1$.
(2) *Local linear independence.* The nonzero functions of $N_{0,k}, \dots, N_{n+k-1,k}$ form a basis for $S_{k,\tau}$. Moreover, if we have a linear combination f and the coefficient of a basis function $N_{i,k}$ is nonzero, then it can be checked that $[t_{i-k}, t_{i+1}] = \operatorname{supp} N_{i,k} \subseteq \operatorname{supp} f$. Then, by Proposition 3.2 of [22], the basis functions are locally linearly independent and so, by Theorem 3.7 of [22], the basis $(N_{0,k}, \dots, N_{n+k-1,k})$ is least supported in the sense explained in that paper.
(3) *Properties at the endpoints.* Any nonzero function $N_{i,k}$, is a $k - m_j$ times continuously differentiable function at each knot $t_j \in [t_{i-k}, t_{i+1}]$ where m_j denotes the number of times t_j appears in the set $\{t_{i-k}, \dots, t_{i+1}\}$. Moreover, at the endpoints of its support we have

$$\begin{aligned} &N_{i,k}^{(\ell)}(t_{i-k}) = 0, \quad \ell = 0, \dots, k - m_{i-k} - 1, \qquad N_{i,k}^{(k-m_{i-k})}(t_{i-k}) > 0, \\ &N_{i,k}^{(\ell)}(t_{i+1}) = 0, \quad \ell = 0, \dots, k - m_{i+1} - 1, \qquad N_{i,k}^{(k-m_{i+1})}(t_{i+1}) \neq 0. \end{aligned} \tag{45}$$

(4) *Relation with the normalized B-basis of* U_k. For the particular knot vector

$$\tau = \{t_k = \cdots = t_0 < t_1 = \cdots = t_{k+1}\},$$

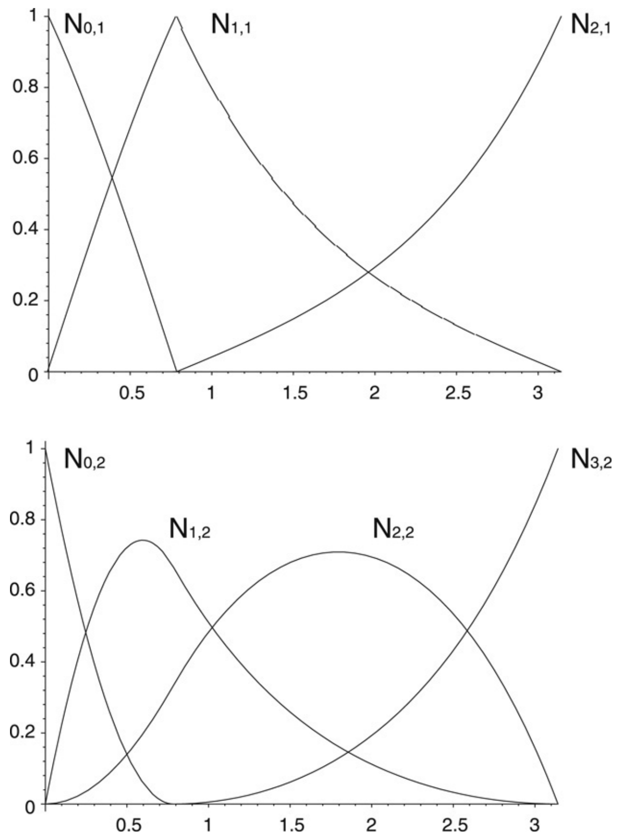

Fig. 2. Generalized B-spline bases of order 1 and 2, for the knot vector $\tau = \{0, 0, \pi/4, 3, 3\}$ with $S_0(t) = \sin t$ and $S_1(t) = \sinh t$.

$S_{k,\tau} = U_k^{(0)}$, where $U_k^{(0)}$ is one of the spaces of (6) and (7) or (7) and the corresponding generalized B-spline basis defined in (44) coincides with the basis defined in (9). Then, if $t_1 - t_0 < z_{S_0}$, then $(N_{0,k}, \dots, N_{k,k})$ is the normalized B-basis of $S_{k,\tau}$.

Fig. 3 illustrates the generalized B-spline bases of S_{1,τ_1}, S_{2,τ_2}, S_{3,τ_3}, S_{4,τ_4} and S_{5,τ_5} obtained with the knot vector $\tau_1 = \{0, 0, 1, 2, 5, 5, 5, 5, 8, 9, 10, 10\}$ for two different cases. In (a) $S_i(t) = \sin t$ for all i; in (b) $S_i(t) = \sinh t$ for all i. For the trigonometric case (a), since $h_2 = h_7 = 3 > z_{S_i'} = \pi/2$, we cannot guarantee the system is a B-basis. Observe the effect of the repeated knots: $t_4 = \cdots = t_7 = 5$ in both cases. In $S_{1,\tau}$, $N_{4,1} \equiv 0$, $N_{5,1} \equiv 0$ and the functions $N_{3,1}$, $N_{6,1}$ are not continuous at $t = 5$. In S_{2,τ_2}, $N_{5,2} \equiv 0$ and the functions $N_{4,2}$, $N_{6,2}$ are not continuous at $t = 5$. In S_{3,τ_3}, there are no identically zero functions and, $N_{5,3}$ and $N_{6,3}$ are not continuous at $t = 5$. In S_{4,τ_4} and S_{5,τ_5}, all the functions are continuous and continuously differentiable, respectively on [0, 10].

Theorem 14. *If $0 \le h_i < z_{S_i'}$ for $i = 0, \dots, n + k - 2$, the nonzero functions of the system $(N_{0,k}, \dots, N_{n+k-1,k})$ defined in (44) form the normalized B-basis of S_{k,τ_k}, for $k > 1$.*

Proof. The nonzero functions of $(N_{0,k}, \dots, N_{n+k-1,k})$ form a basis of S_{k,τ_k}, as stated before.

By Theorem 13, $(N_{0,1}, \dots, N_{n,1})$ is totally positive, and then for any $\tau_0 < \tau_1$ and $i < j$

$$\begin{vmatrix} N_{i,1}(\tau_0) & N_{j,1}(\tau_0) \\ N_{i,1}(\tau_1) & N_{j,1}(\tau_1) \end{vmatrix} \ge 0.$$

Thus we can guarantee that $N_{i,1}/N_{j,1}$ is a decreasing function on the set $\{t \in \mathbf{R} | N_{j,1}(t) \ne 0\}$. Moreover, since the restrictions of the functions of $(N_{0,1}, \dots, N_{n,1})$ to any $[t_i, t_{i+1}]$ with $t_i < t_{i+1}$ are functions in $U_1^{(i)}$ which is, by Theorem 4, an extended Chebyshev space, we can deduce that $N_{i-1,1}/N_{i,1}$ is strictly decreasing on (t_{i-1}, t_{i+1}). Let

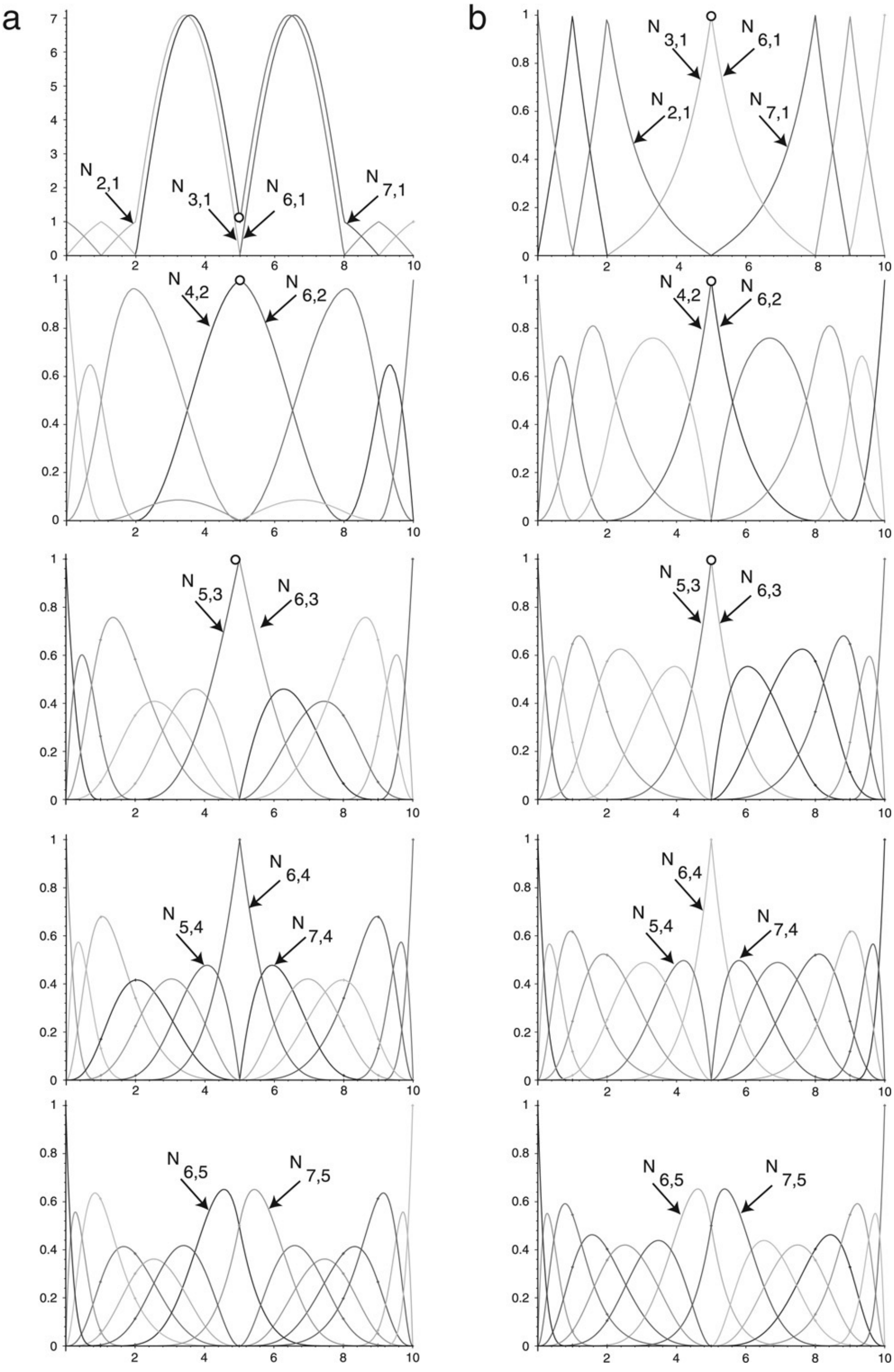

Fig. 3. Generalized B-spline bases of $S_{1,\tau}$, S_{2,τ_2}, S_{3,τ_3}, S_{4,τ_4}, S_{5,τ_5}.

us now consider $(N_{0,2}, \ldots, N_{n+1,2})$. Using the convection $\delta_{-1,1} := 0$ and $\delta_{n+1,1} := 0$, we can write

$$N'_{i,2}(t) = \delta_{i-1,1} N_{i-1,1}(t) - \delta_{i,1} N_{i,1}(t), \quad i = 0, \ldots, n+1, \tag{46}$$

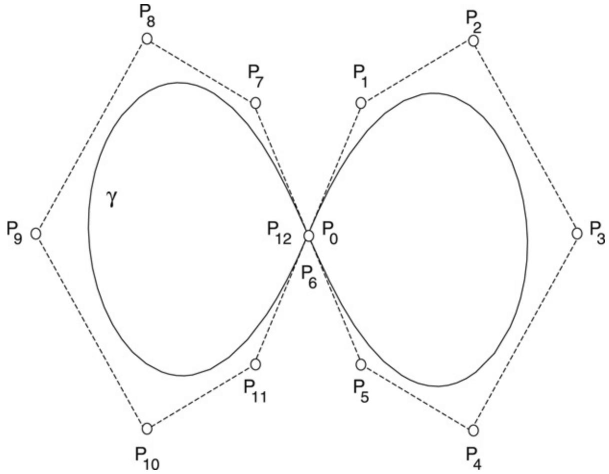

Fig. 4. B-spline curve in S_{4,τ_4}, for $\tau_4 = \{0, 0, 0, 0, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 9, 9, 9, 9\}$.

and, taking into account that $N_{i-1,1}/N_{i,1}$ is strictly decreasing on (t_{i-1}, t_{i+1}), we deduce that $N'_{i,2}(t)$ vanishes at most once on (t_{i-1}, t_{i+1}). Finally, from (44) and the properties at the endpoints (see formulae (45)), we conclude the positivity of $N_{i,2}$ on (t_{i-2}, t_{i+1}).

Since $U_2^{(j)}$, is an extended Chebyshev space (see Theorem 4), S_{2,τ_2} is a space of Chebyshevian splines, and then it is well known that it has a totally positive basis (see Chapter 9 of [23]). It can be proved that this totally positive basis is normalizable because $(N_{0,2}, \dots, N_{n+1,2})$ is normalized. It can also be checked that

$$\lim_{t \to t_{j-2}^+} N_{i,2}(t)/N_{j,2}(t) = 0, \quad \text{and} \quad \lim_{t \to t_{i+1}^-} N_{j,2}(t)/N_{i,2}(t) = 0 \tag{47}$$

whenever $j < i$. Since S_{2,τ_2} has a normalizable totally positive basis, by the implication (ii) implies (i) of Theorem 3.2 of [17], the nonzero functions of $(N_{0,2}, \dots, N_{n+1,2})$ form the normalized B-basis of S_{2,τ_2}. Using the same reasoning as for $(N_{0,2}, \dots, N_{n+1,2})$, we can prove that if $(N_{0,k}, \dots, N_{n+k-1,k})$ is a totally positive system generating S_{k,τ_k}, then the nonzero functions of $(N_{0,k+1}, \dots, N_{n+k,k+1})$ form the normalized B-basis of $S_{k+1,\tau_{k+1}}$, and the result follows for $k \geq 2$. ■

From the previous result, we can derive two additional properties of the generalized B-spline basis. On the one hand, by (47) and the implication (ii) implies (i) of Theorem 3.1 of [19], it has optimal stability properties. On the other hand, since it is totally positive and the basis functions are locally linearly independent (as mentioned previously), when they are continuous the basis satisfies the Schoenberg–Whitney property for the Lagrange interpolation problem by Theorem 3.1 of [24].

Let us now define generalized B-spline curves on $[a, b]$ with $a < b$. For the knot sequence τ_k

$$\tau_k := \{a = t_{-k} = \dots = t_{-1} = t_0 < t_1 \leq \dots \leq t_{n-1} < t_n = t_{n+1} = \dots = t_{n+k} = b\}$$

a *generalized B-spline* curve in $S_{k,\tau}$ can be written as

$$\gamma_k(t) = \sum_{i=0}^{n+k-1} P_i N_{i,k}(t), \quad t \in [a, b], \tag{48}$$

where P_i $(i = 0, \dots, n + k - 1)$ are the control points, i.e., the control polygon of γ_k is $P_0 \cdots P_{n+k-1}$.

Due to the use of the normalized B-basis, generalized B-spline curves have properties similar to those of polynomial B-spline curves (see Fig. 4).

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